MTH 512 H.W-1

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1)
$$\left[\begin{array}{ccc|ccc} 1 & 2 & 6 & 1 & 0 & 0 \\ -1 & 5 & -6 & 0 & 1 & 0 \\ -1 & -2 & -5 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_1+R_3 \to R_3]{R_1+R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 6 & 1 & 0 & 0 \\ 0 & 7 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{R_2}{7} \to R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 6 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/7 & 1/7 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow{-2R_2+R_1 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 6 & 5/7 & -2/7 & 0 \\ 0 & 1 & 0 & 1/7 & 1/7 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-6R_3+R_1 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -37/7 & -2/7 & -6 \\ 0 & 1 & 0 & 1/7 & 1/7 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} -37/7 & -2/7 & -6 \\ 1/7 & 1/7 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$(A^T)^{-1} = (A^{-1})^T = \begin{bmatrix} -\frac{37}{7} & \frac{1}{7} & 1 \\ -\frac{2}{7} & \frac{1}{7} & 0 \\ -6 & 0 & 1 \end{bmatrix}$$

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2) $AX = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$ over $\mathbb{Q}$, we know $X = A^{-1}\begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$

$$X = \begin{bmatrix} -37/7 & -2/7 & -6 \\ \frac{1}{7} & 1/7 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} = 2\begin{bmatrix} -\frac{37}{7} \\ \frac{1}{7} \\ 1 \end{bmatrix} + 0 + 4\begin{bmatrix} -6 \\ 0 \\ 1 \end{bmatrix}, \quad X = \begin{bmatrix} -\frac{242}{7} \\ 2/7 \\ 6 \end{bmatrix}$$

$\therefore$ Solution Set is $\left\{\left(\frac{-242}{7}, \frac{2}{7}, 6\right)\right\}$

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3) Is A nonsingular over $\mathbb{Z}_7$?

No

$$|A| = \begin{vmatrix} 1 & 2 & 6 \\ -1 & 5 & -6 \\ -1 & -2 & -5 \end{vmatrix} = (1)\left[-25-12\right] + 1\left(-10+12\right) - 1\left(-12-30\right)$$

$$= 44 - 37 = 7$$

in $\mathbb{Z}_7$ $\quad 7 \bmod 7 = 0$

det $= 0$,

$\therefore$ A is singular in $\mathbb{Z}_7$

4) $\left[\begin{array}{cccc|c} 5 & 7 & -1 & 2 & 2 \\ -10 & -12 & 2 & -1 & 5 \end{array}\right] \xrightarrow{\frac{1}{5}R_1} \left[\begin{array}{cccc|c} 1 & 7/5 & -1/5 & 2/5 & 2/5 \\ -10 & -12 & 2 & -1 & 5 \end{array}\right]$

$\underset{10R_1 + R_2 \to R_2}{} \left[\begin{array}{cccc|c} 1 & 7/5 & -1/5 & 2/5 & 2/5 \\ 0 & 2 & 0 & 3 & 9 \end{array}\right] \xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{cccc|c} 1 & 7/5 & -1/5 & 2/5 & 2/5 \\ 0 & 1 & 0 & \frac{3}{2} & \frac{9}{2} \end{array}\right]$

$-\frac{7}{5}R_2 + R_1 \to R_1 \quad \left[\begin{array}{cccc|c} 1 & 0 & -1/5 & -17/10 & -59/10 \\ 0 & 1 & 0 & 3/2 & 9/2 \end{array}\right]$

$\therefore$

$$\left.\begin{aligned} X_1 &= \frac{-59}{10} + \frac{1}{5}X_3 + \frac{17}{10}X_4 \\ X_2 &= \frac{-3}{2}X_4 + \frac{9}{2} \end{aligned}\right| \text{ in } \mathbb{Q}$$

in $\mathbb{Z}_{11}$ $\quad \frac{1}{5} = 1\cdot 5^{-1} = 1\cdot 9 = 9$, $\quad \frac{17}{10} = 6\cdot 10^{-1} = 6\cdot 10 = 5$

$\frac{-59}{10} = -4\cdot 10^{-1} = 7\cdot 10 = 4$, $\quad \frac{-3}{2} = -3\cdot 2^{-1} = 8\cdot 6 = 4$

$\frac{9}{2} = 9\cdot 2^{-1} = 9\cdot 6 = 10$

$\therefore$

$$\left.\begin{aligned} X_1 &= 4 + 9X_3 + 5X_4 \\ X_2 &= 4X_4 + 10 \end{aligned}\right\} \text{ in } \mathbb{Z}_{11}$$

Solution Set:

$\{(9X_3 + 5X_4 + 4,\ 4X_4 + 10,\ X_3,\ X_4) \mid X_3, X_4 \in \mathbb{Z}_{11}\}$

No of Solutions $= \|x\| =$ 121 Solutions

5) Prove: $|A^{-1}| = \frac{1}{|A|}$

We know $AA^{-1} = I$, $\therefore |AA^{-1}| = |I| = 1$

$\therefore \quad |A|\,|A^{-1}| = 1, \quad \Rightarrow \boxed{|A^{-1}| = \frac{1}{|A|}}$

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6) Suppose we take 2 colums C_a, C_k of A which are identical
Take the transpose of A, (A^T), and we know $|A^T| = |A|$. Now apply row operations to (A^T), s.t $A^T \xrightarrow{-R_a+R_k \to R_k} B$ (This operation has no effect on $|A^T|$.

$\therefore \quad |A| = |A^T| = |B| = 0$ (since B has a row with all 0's)

7) $A = \begin{bmatrix} 2 & 4 & 1 & 2 \\ 9 & 5 & 6 & 5 \\ 9 & 7 & 4 & 5 \\ 9 & 7 & 10 & 6 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1 \to R_1} \underbrace{\begin{bmatrix} 1 & 2 & \frac{1}{2} & 1 \\ 9 & 5 & 6 & 5 \\ 9 & 7 & 4 & 5 \\ 9 & 7 & 10 & 6 \end{bmatrix}}_{B} \xrightarrow[\substack{-9R_1+R_3 \to R_3 \\ -9R_1+R_4 \to R_4}]{-9R_1+R_2 \to R_2} \begin{bmatrix} 1 & 2 & \frac{1}{2} & 1 \\ 0 & -13 & 3/2 & -4 \\ 0 & -11 & -1/2 & -4 \\ 0 & -11 & 11/2 & -3 \end{bmatrix}$

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$|B| = \frac{1}{2}|A| \Rightarrow |A| = 2|B|$

$|B| = |C| \quad \therefore \quad \underline{|A| = 2|C|}$

$|C| = 1 \begin{vmatrix} -13 & 3/2 & -4 \\ -11 & -1/2 & -4 \\ -11 & 11/2 & -3 \end{vmatrix} = \begin{vmatrix} -13 & \frac{3}{2} & -4 \\ -11 & -\frac{1}{2} & -4 \\ -11 & \frac{11}{2} & -3 \end{vmatrix}$

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$-13 \begin{vmatrix} -\frac{1}{2} & -4 \\ \frac{11}{2} & -3 \end{vmatrix} + 11(-1)^3 \begin{vmatrix} \frac{3}{2} & -4 \\ \frac{11}{2} & -3 \end{vmatrix} - 11 \begin{vmatrix} \frac{3}{2} & -4 \\ -\frac{1}{2} & -4 \end{vmatrix} = -13\left(1.5 + \frac{44}{2}\right) + 11\left(-\frac{9}{2} + \frac{44}{2}\right) - 11(-6+2) = -25$

$\Rightarrow |A| = -50$ in $\mathbb{Q}$ $\quad 11 - (50 \bmod 11)$

$11 - 6 = 5$

$\Rightarrow |A| = 5$ in $\mathbb{Z}_{11}$

$\therefore |3A^T(A^{-1})^2| = |3A^T| \cdot |(A^{-1})^2| = 3^4|A^T| \cdot \left(\frac{1}{|A|}\right)^2 = 3^4 \cdot 5 \cdot \frac{1}{5^2} = 81 \cdot 5^{-1}$

$= 4 \cdot \underset{\substack{\downarrow \\ 9}}{5^{-1}} = 4 \cdot 9 = 3$ in $\mathbb{Z}_{11}$